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THE EFFECTIVE FIELD METHOD IN LINEAR PROBLEMS OF STATICS OF COMPOSITE MEDIA*

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A composite material consisting of a homogeneous matrix containing a random, spatially homogeneous set of ellipsoidal inclusions with different physical and mechanical properties is investigated. Static are studied (such as electric, elastic, stationary temperature, etc.), induced in such a medium by the action of various specified external fields. The problem consists of determining the statistical moments of the random tensor-type functions of the flux density and field strength in the composite material in question. The problem of describing many important physical and mechanical properties of the heterogeneous materials can be reduced to that of solving the above problem.

An approximate method of solution is proposed, based on replacing the local external fields containing single inclusions by the effective field of prescribed structure. In the case of a known effective field which is assumed to vary randomly from one inclusion to the next, the flux density and field intensity tensors in an inhomogeneous medium are restored over this field uniquely. The solution of the stochastic system in question is reduced to construction of the statistical moments of the effective field. It is shown that moments of different orders are connected to each other my means of an infinite sequence of interlinked equations. Introduction of additional assumptions concerning the statistical properties of the effective field, makes it possible to truncate the chain and thus obtain a closed system of equations for several first order moments. Expressions are obtained for the first two statistical moments of the unknown random functions in terms of the corresponding moments of the effective field.

An operator connecting the mathematical expectations of the flux density and field strength tensors in a composite medium (operator of effective properties) is studied. In the general case the operator is nonlocal. It follows therefore that a homogeneous medium used to replace the initial inhomogeneous material in the course of determining the mean values of the unknown fields in terms of the given external field, has the property of spatial dispersion. In the case of sufficiently smooth external fields the effective properties operator can be approximated by a differential operator of finite order. Moreover, for an elastic inhomogeneous medium the equations satisfied by the averaged field potential (displacement vector) coincide, in the first approximation, with the equations of one of the known versions of the couple stress theory of elasticity. The approach proposed here is a development of the method of self-consistent field which was used in /1 − 7/ to construct the effective elastic constants for composite materials.

1. Formulation of the problem. Let us consider a composite material consisting of homogeneous components, namely the matrix and a set of inclusions occupying a system of isolated regions V_k the characteristic functions of which are $V_k(x)$, $k = 1, 2, \ldots$. Let the properties of the medium at any point $x(x_1, x_2, x_3)$ be given by the tensor function

$$c(x) = c_0 + \sum_k c_1^{(k)} V_k(x) \tag{1.1}$$

where c_0 is the tensor of matrix properties and $c_0 + c_1^{(k)}$ is the corresponding tensor for the k-th inclusion. The actual character of the tensor $c(x)$ may vary. In the problem of electric conductivity it becomes a bivalent tensor of the electric conductivity properties of the medium, in the case of a stationary temperature field the components of the tensor $c(x)$ will be the heat conductivity coefficients, and in the problem of elasticity $c(x)$ becomes a tetravalent tensor of the moduli of elasticity.

A system of equations describing a static field with scalar potential u in an inhomogeneous medium will, in a number of important cases, have the form

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$$\operatorname{div} \sigma(x) = q(x), \quad \sigma(x) = c(x) \cdot \varepsilon(x), \quad \varepsilon(x) = \nabla u(x) \tag{1.2}$$

where ∇ denote the grad operation, ε is the field strength, σ is the flux density vector and q is the volume field sources density. If u is a vector potential, then the system (1.2) in which the operator ∇ must be replaced by a symmetrized gradient, describes the displacement vector fields $u(x)$, stress $\sigma(x)$ and deformation $\varepsilon(x)$ tensor fields in an inhomogeneous elastic material. A dot in (1.2) denotes the tensor contraction over one (scalar potentials) or two (vector potential) indices. Without pausing to consider the edge effects we assume that the medium occupies the whole space. We denote by $\sigma_0(x)$ and $\varepsilon_0(x)$ the flux density and external field strength tensors, respectively. Such a field (with potential $u_0(x)$) could exist in a homogeneous medium with properties c_0, with the volume sources and conditions at infinity given.

Let the sets of regions V_k and tensors $c_1^{(k)}$ in (1.1) be random, with all statistical moments of the random function $c(x)$ known. Then the fields $\sigma(x)$ and $\varepsilon(x)$ induced in the inhomogeneous medium by a specified external field will also be random. The main problem dealt with in this paper consists of constructing the statistical moments for these random fields. Let us pass from the differential equations (1.2) to an equivalent system of integral equations for the functions $\sigma(x)$ and $\varepsilon(x)$. The equations can be written in the form /8,9/

$$\sigma(x) = \sigma_0(x) + \int S(x - x') \cdot m(x')\, dx' \tag{1.3}$$

$$\varepsilon(x) = \varepsilon_0(x) + \int K(x - x') \cdot c_0 \cdot m(x')\, dx' \tag{1.4}$$

$$m(x) = \sum_k m^{(k)}(x), \quad m^{(k)}(x) = B_1^{(k)} \cdot \sigma(x) V_k(x) \tag{1.5}$$

$$B_1^{(k)} = (c_0 + c_1^{(k)})^{-1} - c_0^{-1}$$

and the integration here is carried out over the whole space.

The kernels $S(x)$ and $K(x)$ of the integral operators S and **K** in (1.3) and (1.4) are given in terms of the second derivatives of the Green's function $G(x)$ of the homogeneous medium c_0, by the formulas

$$K(x) = -\nabla\nabla G(x), \quad S(x) = c_0 \cdot K(x) \cdot c_0 - c_0 \delta(x) \tag{1.6}$$
$$(\nabla \cdot c_0 \cdot \nabla G(x) = -\delta(x))$$

where $\delta(x)$ is the delta function. From this it follows that $K(x)$ and $S(x)$ are homogeneous generalized third degree functions. The operators S and **K** can be regarded as pseudodifferential /10/, and their expressions $S(k)$ and $K(k) \cdot c_0$ (Fourier transforms of the functions $S(x)$ and $K(x) \cdot c_0$) are homogeneous functions of zero degree in k. The action of such operators, e.g. of the operator S on the square integrable in $\mathbf{R}^3$ functions is described by a relation of the form

$$(\mathrm{S}m)(x) = (2\pi)^{-3} \int S(k) \cdot m(k) \exp[-i(k \cdot x)]\, dk$$

where $m(k)$ is the Fourier transform of the function $m(x)$.

Below we shall assume that the set of inclusions is uniformly distributed over the whole space. If at the same time the external field $\sigma_0(x)$ $(\varepsilon_0(x))$ is a bounded oscillatory function, then the tensors $\sigma(x)$, $\varepsilon(x)$ and $m(x)$ in (1.3) and (1.4) will be non-finite random oscillatory functions. The integrals expressing the action of the operators S and **K** on the realizations of such functions, formally diverge at zero and infinity.

Let us consider a scheme for constructing the formulas of regular representation of the operators S and **K** on the functions of the type shown. The functions can be represented with the accuracy of up to the square integrable in $\mathbf{R}^3$ terms, in the form of exponential series containing, in general, mutually incommensurable wave vectors

$$m(x) = m_0 + \sum_j m_j \exp[i(k_j \cdot x)] \tag{1.7}$$

Here m_0 is the constant component of the function $m(x)$, and the coefficients m_j are such that the series in the above expression converges.

Using the property of contraction we can show that the action of the operators S and **K** on the function $\exp[i(k_j x)]$ can be reduced to multiplying this function by constant multipliers $S(k_j)$ and $K(k_j) \cdot c_0$, respectively. Since $S(k)$ and $K(k)$ are homogeneous functions of zero degree the multipliers are uniquely defined and uniformly bounded for all k $(k_j \neq 0)$. Thus, when the operator S(**K**) acts on the series in (1.7), the latter is transformed into an analogous series with coefficients $S(k_j) \cdot m_j$ $(K(k_j) \cdot c_0 \cdot m_j)$ which converges absolutely, provided that the parent series also converges absolutely.

It remains to determine the action of the operators S and $\mathbf{K}$ on the constant m_0. In the k-space of Fourier transformations this corresponds to multiplying the homogeneous functions $S(k)$ and $K(k)\cdot c_0$ by the delta function concentrated at zero, and this operation is obviously not single-valued. It can be shown that no "natural regularization" of the operators S and $\mathbf{K}$ over the constants exists, and the result depends on which of the external fields, the flux density $\sigma_0(x)$ or field strength $\varepsilon_0(x)$ in the problem is fixed. In particular, if we fix the external flux density, then the following relations hold /5,7/:

$$\int S(x-x')\cdot m_0 dx' = 0, \quad \int K(x-x')\cdot c_0\cdot m_0 dx' = m_0 \tag{1.8}$$

Next we pass to the problem of constructing the statistical moments of the solution of the system (1.3), (1.4).

2. Effective field.

Let us fix one of the typical realizations of the random set of inclusions, and consider any one inclusion with index i. We denote by $\bar{\sigma}_i(x)$ the local external field containing the i-th inclusion. In the region V_i this field is composed of the external field $\sigma_0(x)$ and a field generated by the neighboring inclusions. The expression for $\bar{\sigma}_i(x)$ follows from (1.3) and has the form

$$\bar{\sigma}_i(x) = \sigma_0(x) + \sum_{k\neq i}\int S(x-x')\cdot m^{(k)}(x')\,dx', \quad x\in V_i \tag{2.1}$$

where the functions $m^{(k)}(x)$ are given by (1.5). From the structure of equation (1.3) it follows that the functions $m^{(k)}(x)$ depend only on the value of the field $\bar{\sigma}_k(x)$ in V_k, as well as on the properties and form of the k-th inclusion.

Let us assume that the solution of the problem for a single k-th inclusion in an arbitrary external field is known. This implies that the explicit form of the relation $m^{(k)}(x, \bar{\sigma}_k)$ which can be written as

$$m^{(k)}(x, \bar{\sigma}_k) = (\mathbf{P}_k\bar{\sigma}_k)(x)\,V_k(x) \tag{2.2}$$

is also known. Here the linear operator $\mathbf{P}_k$ is determined from the solution of the problem for an isolated k-th inhomogeneity, and is assumed known. Substituting (2.2) into (2.1) we obtain a system of equations satisfied by the fields $\bar{\sigma}_i(x)$ in a medium with mutually interacting inclusions

$$\bar{\sigma}_i(x) = \sigma_0(x) + \sum_{k\neq i}\int S(x-x')(\mathbf{P}_k\bar{\sigma}_k)(x')\,V_k(x')\,dx' \tag{2.3}$$
$$x\in V_i, \quad i=1,2,\ldots$$

If the functions $\bar{\sigma}_k(x)$ are obtained from the solution of this system, then the fields $\sigma(x)$ and $\varepsilon(x)$ sought will be uniquely determined from the relations (1.3) and (1.4) in which the functions $m^{(k)}(x)$ assume the form (2.2). Thus for the given operators $\mathbf{P}_k$ the functions $\bar{\sigma}_k(x)$ represent the principal unknowns of the problem.

Let $V=\bigcup V_k$ be a region occupied by the inclusions. Fixing an arbitrary point $x_0\in V$ we define the region V_{x_0} by the relation

$$V_{x_0} = \bigcup_{k\neq i} V_k, \quad x_0\in V_i \tag{2.4}$$

and denote the characteristic functions (with argument x) of the regions V and V_{x_0} by $V(x)$ and $V(x_0; x)$, respectively. We introduce the field $\bar{\sigma}(x)$ coinciding in V_k with $\bar{\sigma}_k(x)$, and a linear operator $\mathbf{P}$ such that the following relations hold:

$$(\mathbf{P}\bar{\sigma})(x)\,V(x) = \sum_k(\mathbf{P}_k\bar{\sigma}_k)(x)\,V_k(x) \tag{2.5}$$
$$(\mathbf{P}\bar{\sigma})(x)\,V(x_0;x) = \sum_{k\neq i}(\mathbf{P}_k\bar{\sigma}_k)(x)\,V_k(x), \quad x_0\in V_i$$

Using the notation given we can write the system (2.3) in the form of a single equation for the field $\bar{\sigma}(x)$ in region V

$$\bar{\sigma}(x) = \sigma_0(x) + \int S(x-x')\cdot(\mathbf{P}\bar{\sigma})(x')\,V(x;x')\,dx', \quad x\in V \tag{2.6}$$

If the set of inclusions is random, then $\bar{\sigma}(x)$ is a random function. The problem of constructing the statistical moments $\bar{\sigma}(x)$ reduces to solving the problem of mutual interaction between a number of inclusions, and just admits the exact solution. To make the problem more accessible, we introduce the following simplifying assumptions (hypotheses of the effective field method).

H_1) The field $\bar{\sigma}(x)$ has the same structure in any region occupied by the inclusions. In what follows we shall assume that the dependence of $\bar{\sigma}(x)$ on the coordinates has the form of a polynomial in every region V_k. The degrees of these polynomials are the same for all inclusions and the coefficients vary randomly from one inclusion to the next.

H_2) The values of the random function $\bar{\sigma}(x)$ at the points of the region V_k are statistically independent of the properties of the inclusions occupying the region, and of the geometrical characteristics of the region. The idea behind the hypothesis H_2 is that the local external field in which the arbitrary inclusion is present, is assumed to be weakly dependent on the form and properties of each separate inclusion, but is determined by the integral characteristics of the whole random set of inhomogeneities.

In what follows we shall assume that all inclusions are ellipsoidal. Then, from hypothesis H_1 and solution of the problem of isolated ellipsoidal inhomogeneity in a polynomial external field, it follows /9/ that the field $\sigma(x)$ within the k-th inclusion (k is arbitrary) (and hence the function $m^{(k)}(x, \bar{\sigma}_k)$ in (2.2)) is a polynomial of the same degree as the local external field $\bar{\sigma}_k(x)$. In particular, if the field $\bar{\sigma}(x)$ is assumed constant in the regions V_k, then the operator $\mathbf{P}$ in (2.5) represents a multiplication by the function $P^0(x)$ which is constant in every region V_k, i.e.

$$(\mathbf{P}\bar{\sigma})(x) = P^0(x)\cdot\bar{\sigma}(x), \quad x \in V \tag{2.7}$$

$$P^\circ(x) = P_k^\circ = B_1^{(k)}\cdot(1 - D_k\cdot B_1^{(k)})^{-1}, \quad \bar{\sigma}(x) = \bar{\sigma}_k \tag{2.8}$$

$$\text{when } x \in V_k, \quad k = 1, 2, \ldots$$

Here 1 is a bivalent unit tensor, and the constant tensor D_k is defined by

$$D_k = \frac{1}{4\pi}\int_{\Omega_1} S(a_k^{-1}k)\, d\Omega \tag{2.9}$$

where $S(k)$ is the symbol of the operator S, Ω_1 is the surface of a unit sphere in the k-th space and a_k^{-1} is a linear transformation transforming the ellipsoid V_k into a unit sphere. Substituting (2.7) into (2.6) we arrive at the following expression for the field $\bar{\sigma}(x)$ in V:

$$\bar{\sigma}(x) = \sigma_0(x) + \int S(x - x')\cdot P^\circ(x')\cdot\bar{\sigma}(x')\, V(x; x')\, dx', \quad x \in V \tag{2.10}$$

If the assumption of the constancy of the field $\bar{\sigma}(x)$ in the regions V_k is confirmed, then the solutions of (2.6) and (2.10) coincide.

Let us now assume that the field $\bar{\sigma}(x)$ is linear in V_k occupied by the inclusions (ξ_k is the center of the region V_k)

$$\bar{\sigma}^\alpha(x) = \bar{\sigma}_k^{\ \alpha} + \tau_k^{\alpha\beta}(x - \xi_k)_\beta, \quad x \in V_k, \quad k = 1, 2, \ldots \tag{2.11}$$

Here and henceforth the co- and contravariant components of the tensors on an arbitrary oblique -angled basis will be identified by the lower and upper Greek indices, respectively. The same lower and upper indices will denote summation. Since the linear external field induces a linear field inside every ellipsoidal heterogeneity /9/, therefore the operator $\mathbf{P}$ appearing in (2.6) acts on $\bar{\sigma}(x)$ according to the formula

$$(\mathbf{P}\bar{\sigma})^\alpha(x) = (P_k^\circ)_\beta^{\ \alpha}\bar{\sigma}_k^{\ \beta} + (Q_k)_{\beta\nu}^{\alpha\mu}\tau_k^{\beta\nu}(x - \xi_k)_\mu, \quad x \in V_k, \quad k = 1, 2, \ldots \tag{2.12}$$

Here P_k° has the form (2.8) and the constant tensor Q_k is given, in the absence of the volume sources of external field, in the form

$$(Q_k)_{\beta\lambda}^{\alpha\mu} = (B_1^{(k)})_\beta^\nu R_{\nu\lambda}^{\alpha\mu}, \quad (R^{-1})_{\nu\lambda}^{\alpha\mu} = I_{\nu\lambda}^{\alpha\mu} - (\Pi_k)_{\nu\rho}^{\alpha\mu}(B_1^{(k)})_\lambda^\rho$$

$$(\Pi_k)_{\alpha\lambda}^{\beta\mu} = \frac{3}{4\pi}\int_{\Omega_1}(a_k^{-1}k)_\alpha(a_k k)_\lambda S^{\beta\mu}(a_k^{-1}k)\, d\Omega \tag{2.13}$$

where I is a unit four-valent tensor and Ω_1 denotes, as in (2.9), the surface of a unit sphere in the k-space. Taking into account the fact that the constant tensors $\bar{\sigma}_k$ and τ_k in (2.11) can be expressed in terms of the linear tensors in the region V_k and the field $\bar{\sigma}(x)$

$$\bar{\sigma}_k^{\ \alpha} = \bar{\sigma}^\alpha(x) - (\nabla^\beta\bar{\sigma}^\alpha(x))(x - \xi_k)_\beta, \quad \tau_k^{\alpha\beta} = \nabla^\beta\bar{\sigma}^\alpha(x), \quad x \in V_k \tag{2.14}$$

we conclude that the equation (2.6) for $\bar{\sigma}(x)$ will become, assuming that the hypothesis (2.11) holds,

$$\bar{\sigma}^\alpha(x) = \bar{\sigma}_0^{\ \alpha}(x) + \int S^{\alpha\beta}(x - x')\,[P_{\beta\lambda}^\circ(x')\bar{\sigma}^\lambda(x') + P_{\beta\mu\lambda\nu}^1(x')(\nabla^\mu\bar{\sigma}^\lambda(x'))H^\nu(x')]\, V(x; x')\, dx', \quad x \in V \tag{2.15}$$

The functions $P^1(x)$ and $H(x)$ are defined in every region V_k by the relations ($\delta_{\alpha\beta}$ is the Kronecker delta)

$$P^1_{\alpha\beta\lambda\mu}(x) = (Q_k)_{\alpha\beta\lambda\mu} - (P_s{}^\circ)_{\alpha\beta}\,\delta_{\lambda\mu};$$
$$H^\nu(x) = H_k{}^\nu(x) = (x - \xi_k)^\nu, \quad x \in V_k, \quad k = 1, 2, \ldots \tag{2.16}$$

and equation (2.15) is already integrodifferential with respect to $\bar\sigma(x)$.

Approximating the field $\bar\sigma(x)$ in the regions V_k with the second degree polynomials we arrive, in the same manner, at an integrodifferential equation containing second order derivatives in $\bar\sigma(x)$. We shall continue to call effective the field $\bar\sigma(x)$ satisfying an equation of the type (2.10), (2.15) or other analogous expressions. We shall also call (2.10) a zero order equation and (2.15) a first order equation for the effective field.

3. General scheme for constructing the statistical moments of the solution.

We first consider a problem of constructing the statistical moments of the effective field $\bar\sigma(x)$. Let us denote by $\bar\sigma^n(x_1, x_2, \ldots, x_n)$ the n-point moment of effective field representing the average value of the tensor product $\bar\sigma^{\alpha_1}(x_1)\bar\sigma^{\alpha_2}(x_2)\ldots\bar\sigma^{\alpha_n}(x_n)$, under the condition that the points $x_1, x_2, \ldots, x_n$ lie in the region V occupied by the inclusions. In particular, the mathematical expectation and two-point moment of effective field represent conditional averages of the form

$$(\bar\sigma^1)^\alpha(x) = \langle\bar\sigma^\alpha(x)\,|\,x\rangle, \quad (\bar\sigma^2)^{\alpha\beta}(x_1, x_2) = \langle\bar\sigma^\alpha(x_1)\bar\sigma^\beta(x_2)\,|\,x_1, x_2\rangle \tag{3.1}$$

where $\langle\cdot\,|\,x_1, x_2, \ldots, x_n\rangle$ denotes a mean value under the condition that $x_1, x_2, \ldots, x_n \in V$.

At this point we shall restrict ourselves to considering the zero order equations for effective field (2.10). We construct the mathematical expectation $\bar\sigma^1(x)$ by averaging both sides of (2.10) over the ensemble of random set of inclusions under the condition that $x \in V$:

$$\langle\bar\sigma(x)|\,x\rangle = \sigma_0(x) + \int S(x - x')\cdot\langle P^0(x')\cdot\bar\sigma(x')\,V(x;x')|\,x\rangle\,dx \tag{3.2}$$

The point x is assumed fixed, therefore $S(x - x')$ is a determinate kernel. Using the hypothesis H_2 of Sect.2 stating that the field $\bar\sigma(x)$ is statistically independent in the region V_k of the geometrical characteristics of the latter and the properties of the k-th inclusion, we can write the mean appearing in the integrand of (3.2) in the form of the following product:

$$\langle P^0(x')\cdot\bar\sigma(x')\,V(x;x')\,|\,x\rangle = \langle P^\circ(x')\,V(x;x')\,|\,x\rangle\cdot \langle\bar\sigma(x')\,|\,x';x\rangle \tag{3.3}$$

where $\langle\bar\sigma(x')\,|\,x';x\rangle$ denotes an average value under the condition that $x' \in V$, $x \in V_{x'}$, which is clearly different from $\bar\sigma^1(x)$. If the properties of the inclusions are statistically independent of the geometry and relative distribution of the regions V_k, then the first right comultiplier in (3.3) can be written in the form

$$\langle P^\circ(x')\,V(x;x')\,|\,x\rangle = P^\circ\psi(x, x'), \quad P^\circ = \langle P^\circ(x)\,|\,x\rangle = \langle P_k^\circ\rangle_k \tag{3.4}$$

The right-hand side expression in the last equation depicts the average value of the tensor P_k° of the form (2.8) over the ensemble of inclusions. The scalar function $\psi(x, x')$ in (3.4) is a conditional mean of the form

$$\psi(x, x') = \langle V(x;x')\,|\,x\rangle \tag{3.5}$$

By virtue of the definition (2.4) of the region V_x the above function is equal to zero when $x = x'$. If the set of inclusions is spatially homogeneous, then ψ depends only on the difference $x - x'$. When $|x - x'| \to \infty$, the correlation in the distribution of inclusions vanishes and we have

$$\langle V(x;x')\,|\,x\rangle \to \langle V(x')\rangle = p$$

where p denotes the concentration of the inclusions. For an isotropic set of inclusions $\psi(x)$ depends only on $|x|$.

An example of a function satisfying the above conditions is

$$\psi(x) = p\,(1 - e^{-|x|/\rho}) \tag{3.6}$$

where the parameter ρ denotes the radius of correlation of the random set of inclusions. A conditional mean $\psi(x, x')$ can be constructed for concrete stochastic sets of inclusions using the methods of geometrical theory of probability /11/. Below we shall assume that the form of the function ψ and of the more complicated conditional means of the function $V(x;x')$ are both known.

We construct the second right comultiplier in (3.3) $\langle\bar\sigma(x')\,|\,x';x\rangle$ by averaging both sides of the equation (2.10) with condition $x \in V$, $x_1 \in V_x$, and using once again the hypothesis H_2. This yields the expressions for the means $\langle\bar\sigma(x)\,|\,x\rangle$ and $\langle\bar\sigma(x)\,|\,x;x_1\rangle$ in the form

$$\langle\bar{\sigma}(x)\mid x\rangle = \sigma_0(x) + \int S(x-x')\cdot P^\circ\cdot\langle\bar{\sigma}(x')\mid x'; x\rangle\,\psi(x,x')dx' \tag{3.7}$$

$$\langle\bar{\sigma}(x)\mid x; x_1\rangle = \sigma_0(x) + \int S(x-x')\cdot P^\circ\cdot\langle\bar{\sigma}(x')\mid x'; x, x_1\rangle\cdot \langle V(x;x')\mid x;\ x_1\rangle dx' \tag{3.8}$$

where $\langle\bar{\sigma}(x')\mid x'; x, x_1\rangle$ is a mean under the condition $x'\in V,\ x\in V_x,\ x_1\in V_x$, different from $\langle\bar{\sigma}(x')\mid x'; x\rangle$. Thus we obtain a sequence of equations for the conditional means of the function $\bar{\sigma}(x)$. To close this sequence we must bring in additional assumptions concerning the statistical properties of the effective field. The simplest assumption is represented by an analog of the so-called "quasicrystalline approximation" /12,13/

$$\langle\bar{\sigma}(x')\mid x'; x\rangle = \langle\bar{\sigma}(x')\mid x'\rangle = \bar{\sigma}^1(x') \tag{3.9}$$

We assume here that the mean value of the effective field coincides at the point x' with the value averaged over the set of heterogeneities for which the point x lies within one of the inclusions. This, together with (3.7), yields a closed equation for the mathematical expectation of the effective field

$$\bar{\sigma}^1(x) = \sigma_0(x) + \int S_\psi(x-x')\cdot P^\circ\cdot\bar{\sigma}^1(x')\,dx';\quad S_\psi(x) = S(x)\,\psi(x) \tag{3.10}$$

Solving this equation for $\bar{\sigma}^1(x)$ we obtain

$$\bar{\sigma}^1(x) = (\Lambda\sigma_0)(x) \tag{3.11}$$

where Λ is a pseudodifferential operator the symbol of which is of the form ($S_\psi(k)$ is the Fourier transform of $S_\psi(x)$)

$$\Lambda(k) = (1 - S_\psi(k)\cdot P^\circ)^{-1} \tag{3.12}$$

The following approximation for $\bar{\sigma}^1(x)$ is obtained by truncating the chain in (3.8), with help of the assumption

$$\langle\bar{\sigma}(x')\mid x'; x, x_1\rangle = \langle\bar{\sigma}(x')\mid x'; x_1\rangle = \Phi(x', x_1) \tag{3.13}$$

The function $\Phi(x', x_1)$ represents the mean value of the field $\bar{\sigma}$ at the point $x'\in V$ under the condition that an inclusion is present at the point $x_1\in V_{x'}$. The function characterizes the pair-wise relationship in a system of interacting inclusions. Clearly $\Phi(x', x_1)\to\bar{\sigma}^1(x')$ as $|x'-x_1|\to\infty$. Equation for $\Phi(x; x_1)$ follows from (3.8), (3.13) and has the form

$$\Phi(x, x_1) = \sigma_0(x) + \int S(x-x')\cdot P^\circ\cdot\Phi(x', x_1)\times\langle V(x;x')\mid x; x_1\rangle dx' \tag{3.14}$$

To construct the second moment of effective field $\bar{\sigma}^2(x_1, x_2)$ we tensor-multiply both parts of (2.10) by $\bar{\sigma}(x_2)$ and average the result under the condition that $x_1, x_2\in V$

$$\bar{\sigma}^2(x_1,x_2) = \sigma_0(x_1)\langle\bar{\sigma}(x_2)\mid x_1, x_2\rangle + \int S(x_1-x')\langle P^\circ(x')\cdot\bar{\sigma}(x')\,\bar{\sigma}(x_2)\,V(x_1;x')\mid x_1, x_2\rangle\,dx' \tag{3.15}$$

The averaged term in the integrand can be written, by virtue of the hypothesis H_2, in the form

$$\langle P^\circ(x')\cdot\bar{\sigma}(x')\,\bar{\sigma}(x_2)\,V(x_1;x')\mid x_1, x_2\rangle = P^\circ\cdot\langle\bar{\sigma}(x')\,\bar{\sigma}(x_2)\mid x', x_2; x_1\rangle\langle V(x_1;x')\mid x_1, x_2\rangle$$

Using now an assumption of the type (3.9), (3.13)

$$\langle\bar{\sigma}(x')\,\bar{\sigma}(x_2)\mid x', x_2; x_1\rangle = \langle\bar{\sigma}(x')\,\bar{\sigma}(x_2)\mid x', x_2\rangle = \bar{\sigma}^2(x', x_2) \tag{3.16}$$

we obtain, from (3.15), the following closed equation for $\bar{\sigma}^2(x_1, x_2)$:

$$\bar{\sigma}^2(x_1,x_2) = \sigma_0(x_1)\,\Phi(x_2,x_1) + \int S(x_1-x')\cdot P^\circ\cdot\bar{\sigma}^2(x',x_2)\cdot\langle V(x_1;x')\mid x_1,\ x_2\rangle dx' \tag{3.17}$$

where $\Phi(x_1, x_2)$ is the solution of (3.14). The route for constructing further approximations for $\bar{\sigma}^2(x_1, x_2)$ following the approach given above, is obvious.

Let us now pass to computing the statistical moments of the fields $\sigma(x)$ and $\varepsilon(x)$ in an inhomogeneous medium. If the field $\bar{\sigma}(x)$ is approximated by a constant in every inclusion, then the expressions (1.3), (1.4) for $\sigma(x)$ and $\varepsilon(x)$ will become, by virtue of (2.5) and (2.7),

$$\sigma(x) = \sigma_0(x) + \int S(x-x')\cdot P^\circ(x')\cdot\bar{\sigma}(x')\,V(x')\,dx' \tag{3.18}$$

$$\varepsilon(x) = \varepsilon_0(x) + \int K(x-x')\cdot c_0\cdot P^\circ(x')\cdot\bar{\sigma}(x')\,V(x')\,dx' \tag{3.19}$$

Averaging these relations over the ensemble of the random set of inclusions and remembering that by virtue of the hypothesis H_2

$$\langle P^\circ(x')\cdot\bar{\sigma}(x')\,V(x')\rangle = \langle P^\circ(x')\,V(x')\rangle\cdot\bar{\sigma}^1(x') = pP^\circ\cdot\bar{\sigma}^1(x')$$

we obtain

$$\langle\sigma(x)\rangle = \sigma_0(x) + p\int S(x-x')\cdot P^\circ\cdot\bar{\sigma}^1(x')\,dx' \qquad (3.20)$$
$$\langle\varepsilon(x)\rangle = \varepsilon_0(x) + p\int K(x-x')\cdot c_0\cdot P^\circ\cdot\bar{\sigma}^1(x')\,dx'$$

Next we write the expression for the second moment of the field $\sigma(x)$ in a composite medium, in terms of the conditional moments of the effective field. From (3.18) we have

$$\begin{aligned}\langle\sigma^\alpha(x_1)\sigma^\beta(x_2)\rangle = {} & \sigma_0{}^\alpha(x_1)\sigma_0{}^\beta(x_2) + \\ & p\int S^{\alpha\lambda}(x_1-x')\,\overset{\circ}{P}_{\lambda\mu}\bar{\sigma}^{1\mu}(x')\,dx'\sigma_0{}^\beta(x_2) + \\ & p\sigma_0{}^\alpha(x_1)\int S^{\beta\lambda}(x_2-x')\,\overset{\circ}{P}_{\lambda\mu}\bar{\sigma}^{1\mu}(x')\,dx' + \\ & \int S^{\alpha\lambda}(x_1-x')\,\overset{\circ}{P}_{\lambda\nu}\,dx'\int S^{\beta\mu}(x_2-x'')\,\overset{\circ}{P}_{\mu\rho}\bar{\sigma}^{2\nu\rho}(x',x'')\times \\ & \langle V(x')V(x'')\rangle\,dx''\end{aligned} \qquad (3.21)$$

where we used the hypothesis H_2 of Sect. 2. Similarly, the second moment of the random field $\varepsilon(x)$ can be expressed in terms of the first two conditional moments of the effective field. Thus in order to find the first two statistical moments of the fields $\sigma(x)$ and $\varepsilon(x)$ in a composite medium, we must solve the equations of the type (3.10), (3.17) for $\bar{\sigma}^1$ and $\bar{\sigma}^2$ and compute the integrals in (3.20) and (3.21). The scheme can also be used for computing higher order moments of the fields $\sigma(x)$ and $\varepsilon(x)$.

4. Operator of effective properties.

We introduce the operator $\mathbf{C}_*$ connecting the mathematical expectations of the flux density and field strength tensors in a composite medium

$$\sigma^*(x) = (\mathbf{C}_*\varepsilon^*)(x), \quad \sigma^*(x) = \langle\sigma(x)\rangle, \quad \varepsilon^*(x) = \langle\varepsilon(x)\rangle \qquad (4.1)$$

From (3.20) and (3.11) it follows that the pseudodifferential operator $\mathbf{C}_*$ has the form

$$C_*(k) = (1 + pS(k)\cdot P^\circ\cdot\Lambda(k))\cdot(B_0 + pK(k)\cdot c_0\cdot P^\circ\cdot\Lambda(k))^{-1} \quad (B_0 = c_0{}^{-1}) \qquad (4.2)$$

In the general case the relation connecting σ^* with ε^* will be nonlocal, since $\mathbf{C}_*$ is a contraction operator with generalized function $C_*(x)$ which has a singular and a regular term. The case of a homogeneous external field σ_0 represents an exception. From (3.11), (3.12) it follows that the operator Λ represents a multiplication by a constant tensor

$$\Lambda_0 = [1 - \int S(x)\,\psi(x)\,dx\cdot P^\circ]^{-1} \qquad (4.3)$$

For an isotropic set of inclusions we have $\psi(x) = \psi(|x|)$, and in this case the integral can be written, with help of (1.8) and regularization of the generalized function $S(x)$ /9,10/, in the form

$$\int S(x)\,\psi(x)\,dx = -pD_0$$

where the constant tensor D_0 has the form (2.9) when $a_k = 1$. From (3.20) and (1.8) it follows that for a constant external field σ^* and ε^* the constant tensors are connected by the relation

$$\sigma^* = C_*\cdot\varepsilon^*, \quad C_*{}^{-1} = B_0 + pP^\circ\cdot(1 + pD_0\cdot P^\circ)^{-1} \qquad (4.4)$$

where C_* is the tensor of effective constants of the composite medium. An analogous expression was obtained for C_* in /5/ for the case of an elastic problem (tensor of effective moduli of elasticity of the composite medium), and particular forms of this expression were obtained in /1,3,4/. The effective moduli of elasticity of a medium with a random set of cracks were obtained using the proposed scheme in /2 – 4.7/ and the effective electric conductivity coefficients in /14/.

Let us expand the right-hand side of the expansion (4.2) for $C_*(k)$ into a series in terms of the concentration of inclusions p. Restricting ourselves to terms of order of p^2, we have

$$C_*(k) = c_0 - pc_0\cdot P^\circ\cdot c_0 + p^2c_0\cdot P^\circ\cdot(c_0 + T(k))\cdot P^\circ\cdot c_0 + \cdots \qquad (4.5)$$
$$T(k) = \int S(x)\,(1 - p^{-1}\psi(x))\exp[i(k\cdot x)]\,dx$$

For $\psi(x)$ given by (3.6) the function

$$T(k) = \int S(x)\exp\left[-\frac{|x|}{\rho} + i(k\cdot x)\right]dx$$

It can be shown that function $T(k)$ is analytic in the neighborhood of zero, and the first terms of its expansion in a power series have the form

$$T^{\alpha\beta}(k) = D_0^{\alpha\beta} + {}^1\!/_2\rho^2\,(D_0^{\alpha\beta}\delta_{\lambda\mu} - 3\Pi_{0\lambda\mu}^{\alpha\beta})\,(ik^\lambda)\,(ik^\mu) + \cdots \qquad (4.6)$$

where the tensors D_0 and Π_0 are given by (2.9) and (2.12), respectively, for $a_k = 1$. In the first approximation (for sufficiently smooth external fields) the symbol $C_*(k)$ can be approximated by the expression (4.5) with $T(k)$ of the form (4.6). We introduce the averaged field potential u^* in an inhomogeneous medium, connected with the tensor $\varepsilon^*(x)$ by the relation $\varepsilon^*(x) = \nabla u^*(x)$. From (4.1), (4.2) we obtain the following expression for the potential $u^*(x)$:

$$\operatorname{div}(\mathbf{C}_*\nabla u^*) = q \tag{4.7}$$

where $\mathbf{C}_*$ is a pseudodifferential operator defined by (4.2). Since the operator $\mathbf{C}_*$ is nonlocal, the potential u^* described a field in a certain homogeneous medium with spatial dispersion. This medium replace the real inhomogeneous material in the course of computing the mean values of the functions $\sigma(x)$ and $\varepsilon(x)$ over the given external field.

If we approximate the operator $\mathbf{C}_*$ with the expressions (4.5) and (4.6), then (4.7) will become a partial differential equation in $u^*(x)$. For a composite material consisting of an isotropic matrix and isotropic spherical inclusions, the equation will become (Δ is the Laplace operator and c_{*1}, c_{*2} are scalar coefficients)

$$\begin{gathered}(c_{*1}\Delta - c_{*2}\Delta^2)\, u^*(x) = q(x) \\ c_{*1} = c_0\left(1 - pg + \frac{1}{3}p^2g^2\right), \quad c_{*2} = \frac{2}{15}\rho^2p^2c_0g^2, \\ g = \frac{3c_0B_1}{3+2c_0B_1}, \quad B_1 = \langle B_1^{(k)}\rangle_k\end{gathered} \tag{4.8}$$

In the case of the elastic problem, an analogous expression for the averaged displacement vector will basically coincide with the system of equations of the couple stress theory of elasticity for a medium with restricted rotation /15/. The part of the parameter with dimension of length which is characteristic for the couple stress theory, is played in this case by the correlation radius ρ of the random set of inclusions. We note that the previous expressions for $\mathbf{C}_*$ did not include another characteristic parameter of the problem, namely the mean size of the inclusions. The dependence of C_* on this parameter can be established by realizing the proposed scheme for constructing the operator C_* with help of the first order equation (2.15) for effective field. In this case the analog of the expansion (4.5), (4.6) for $C_*(k)$ will assume, in the case of an isotropic matrix and isotropic (spherical) inclusions, the form

$$C_*^{\alpha\beta}(k) = c_{*1}\delta^{\alpha\beta} - \frac{1}{2}c_{*2}\left(k^2\delta^{\alpha\beta} + 3(ik^\alpha)(ik^\beta)\right)$$

where the coefficient c_{*1} coincides with that given by (4.8) and c_{*2} depends on two dimensional parameters, namely on the correlation radius ρ of the random set of inclusions and their mean radius a

$$c_{*2} = \frac{2}{15}p^2c_0g^2\left(\rho^2 + \frac{2}{15}\,\frac{a^2c_0B_1}{5+3c_0B_1}\right)$$

Let us note the principal characteristic features of the method. The method falls between the self-consistent field (SCF) method and the smoothing method /16/. The proposed method and one of the more widely used variants of SCF both introduce a local external field $\bar{\sigma}$ for every inclusion and assume that the field has the same structure over the regions V_k (hypothesis H_1). However, in the traditional SCF method the field $\bar{\sigma}$ is assumed constant and equal for all inclusions. In the present method the field $\bar{\sigma}$ can: 1°- be chosen differing from the constant value in the regions V_k and, 2°- it can be assumed to vary randomly from one inclusion to the next. Moreover, in constructing the closed equations for the statistical moments $\bar{\sigma}(x)$ we use a procedure consisting of disconnecting the complex averages, which is associated with the smoothing method. Conceptually, the present method resembles that used in /12,13,17/ for solving a stationary problem of scattering scalar waves by point scatterers.

The error incurred in the first approximation of the method in the problem of computing the effective constants for inhomogeneous materials was investigated in /5,6,7/ by comparing them with experimental data and exact solutions obtained for regular composites. The comparison shows that the proposed method yields correct values for the effective constants of the composites over a wide range of variation in the concentration, form and the properties of the inclusions. The possibilities offered by the method are still not completely exhausted. The method can be used to obtain quantitative descriptions of the nonlocal properties of a composite material and to find the expressions for the statistical moments for solutions of, generally speaking, any order.

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